A VANISHING THEOREM FOR CHARACTERISTIC CLASSES OF ODD-DIMENSIONAL MANIFOLD BUNDLES

JOHANNES EBERT

Abstract. We show how the Atiyah-Singer family index theorem for both, usual and self-adjoint elliptic operators fits naturally into the framework of the Madsen-Tillmann-Weiss spectra. Our main theorem concerns bundles of odd-dimensional manifolds. Using completely functional-analytic methods, we show that for any smooth proper oriented fibre bundle $E \to X$ with odd-dimensional fibres, the family index $\mathrm{ind}(B) \in K^1(X)$ of the odd signature operator is trivial. The Atiyah-Singer theorem allows us to draw a topological conclusion: the generalized Madsen-Tillmann-Weiss map $\alpha : B\operatorname{Diff}^+(M^{2m-1}) \to \Omega^\infty \mathrm{MTSO}(2m-1)$ kills the Hirzebruch $\mathcal{L}$-class in rational cohomology. If $m = 2$, this means that α induces the zero map in rational cohomology. In particular, the three-dimensional analogue of the Madsen-Weiss theorem is wrong. For 3-manifolds M, we also prove the triviality of $\alpha : B\operatorname{Diff}^+(M) \to \mathrm{MTSO}(3)$ in mod p cohomology in many cases. We show an appropriate version of these results for manifold bundles with boundary.

Contents

1. Introduction and statement of results

One of the greatest achievements of algebraic topology in the last decade are the two proofs of Mumford's conjecture on the homology of the stable mapping class group by Madsen and Weiss [30] and by Galatius, Madsen, Tillmann and Weiss [20]. The Pontrjagin-Thom construction is crucial for both proofs; it provides a map from the classifying space of the diffeomorphism group of a compact surface to the infinite loop space $\Omega^\infty \mathrm{MTSO}(2)$ of the Madsen-Tillmann-Weiss spectrum, in other words the Thom spectrum of the inverse of the universal complex line bundle.

The proof in [20] consists of two parts. One part (essentially due to Tillmann [40]), exclusively applies to 2-dimensional manifolds, because it relies on two deep

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results of surface theory (the Harer-Ivanov homological stability theorem and the Earle-Eells theorem on the contractibility of the components of the diffeomorphism group of surfaces of negative Euler number). The other part of the proof, however, is valid for manifolds of arbitrary dimension and with general "tangential structures" and provides a vast generalization of the classical Pontrjagin-Thom theorem relating bordism theory of smooth manifolds and stable homotopy.

Given an oriented (we will ignore more general tangential structures throughout the present paper) closed manifold M of dimension n, there exists a map

$$\alpha_{E_M} : B\operatorname{Diff}^+(M) \to \Omega^\infty \mathrm{MTSO}(n), \tag{1.0.1}$$

where $\mathrm{MTSO}(n)$ denotes the Thom spectrum of the inverse of the universal n-dimensional oriented vector bundle. Let Cob_n^+ be the oriented n-dimensional cobordism category: objects are closed $(n-1)$-dimensional manifolds, morphisms are oriented cobordisms and composition is given by gluing cobordisms. With a suitable topology on object and morphism spaces, Cob_n^+ becomes a topological category. The maps α from 1.0.1 assemble to a map

$$\alpha^{GMTW} : \Omega B\,\mathrm{Cob}_n^+ \to \Omega^\infty \mathrm{MTSO}(n), \tag{1.0.2}$$

and the main result of [20] states that α^{GMTW} is a homotopy equivalence. Moreover, for any closed n-manifold M, there is a tautological map $\Phi_M : B\operatorname{Diff}^+(M) \to \Omega B\,\mathrm{Cob}_n^+$ and $\alpha^{GMTW} \circ \Phi_M = \alpha_{E_M}$.

The exclusive result for two-dimensional manifolds is that when M is a closed connected oriented surface of genus g, then Φ_M induces an isomorphism on integral homology groups of degrees $* \leq g/2 - 1$. Both theorems together provide an isomorphism of the homology of $B\operatorname{Diff}^+(M)$ and $\Omega^\infty \mathrm{MTSO}(2)$ (in that range of degrees).

In this paper, we study the map α_{E_M} (or, equivalently, Φ_M) when M is an oriented closed manifold of *odd* dimension. It turns out that α_{E_M} fails to be an isomorphism in homology in any range and that no clue about the homology of $B\operatorname{Diff}^+(M)$ can be derived from the study of α_{E_M}. This seems to be an unsatisfactory state of affairs and therefore we attempt to arouse the reader's curiosity by the following remark:

Even if the map α fails to be an "equivalence" of some kind, it still contains interesting information about $B\operatorname{Diff}^+(M)$. Any cohomology class of $\Omega^\infty \mathrm{MTSO}(n)$ (in an arbitrary generalized cohomology theory) yields, via α_{E_M}, a cohomology class of $B\operatorname{Diff}^+(M)$, also known as a characteristic class of smooth oriented M-bundles. One should think of these characteristic classes as "universal" classes in the sense that they are defined for *all* oriented n-manifolds and are defined using only the *local* structure of the manifold.

Examples are the *generalized MMM-classes* (this is the abbreviation of the names Mumford, Miller, Morita)

$$f_!(c(T_v E)) \in H^{*-n}(B; R),$$

where $f : E \to B$ is a smooth oriented fibre bundle with vertical tangent bundle T_vE, R is a ring and $c \in H^*(BSO(n); R)$ is a characteristic class of oriented vector bundles. The generalized MMM-classes come from spectrum cohomology classes of $\mathrm{MTSO}(n)$.

Other examples come from index theory of elliptic operators. Any sufficiently natural elliptic differential operator on oriented n-manifolds defines a characteristic class in K^0 (namely, the family index). Likewise, a natural self-adjoint elliptic operator has a family index in K^{-1} and so it defines a characteristic class in K^{-1}. An application of the Atiyah-Singer Index theorem shows that these index-theoretic classes also come from $\mathrm{MTSO}(n)$.

On any closed oriented Riemannian manifold of odd dimension, there is the *odd signature operator* $D : \mathcal{A}^{ev}(M) \to \mathcal{A}^{ev}(M)$ on forms of even degree. It is self-adjoint, elliptic and its kernel is the space of harmonic form of even degree, which is isomorphic to $H^{ev}(M; \mathbb{C})$. Given any smooth oriented M-bundle $f : E \to B$ we can choose a Riemannian metric on the fibres and study the induced family of elliptic self-adjoint operators. Here is the central result of the present paper.

Theorem 1.0.3. *The family index of the odd signature operator on an oriented bundle $E \to B$ with odd-dimensional fibres is trivial,* $\mathrm{ind}(D) = 0 \in K^1(B)$.

The proof of this result is entirely *analytic*; it is based on the fact that the kernel dimension of D is constant. Therefore the Atiyah-Singer index theorem allows us to draw topological conclusions from Theorem 1.0.3. Here is one of them:

Theorem 1.0.4. *For any closed oriented $2m-1$-dimensional manifold M, the Madsen-Tillmann-Weiss map* $\Sigma^\infty(B\,\mathrm{Diff}^+(M))_+ \to \mathrm{MTSO}(2m-1)$ *kills the Hirzebruch $\mathcal{L}$-class* $\mathrm{th}_{-L_{2m-1}}\mathcal{L} \in H^{4*-2m+1}(\mathrm{MTSO}(2m-1); \mathbb{Q})$.

In particular, for any oriented smooth fibre bundle $f : E \to B$ with fibre M, the generalized MMM-class $f_!(\mathcal{L}(T_vE)) \in H^*(B; \mathbb{Q})$ is trivial.

The precise meaning of this theorem will be clarified in the main text. If $m = 2$, Theorem 1.0.4 implies:

Corollary 1.0.5. *If* $\dim M = 3$, *the Madsen-Tillmann-Weiss map* $\alpha : B\,\mathrm{Diff}^+(M) \to \Omega^\infty\,\mathrm{MTSO}(3)$ *is trivial in rational cohomology (in positive degrees).*

This is an amusing result. Recently, Hatcher and Wahl [22] showed an analogue of the Harer-Ivanov homological stability for mapping class groups of 3-manifolds. Moreover, for large classes 3-dimensional manifolds, it is known that the components of the diffeomorphism group are contractible (but that tends to become wrong after stabilization). One might be tempted to think that these results helps to make the proof of the analogue of the Mumford conjecture valid, leading to a description of the stable homology of mapping class groups of 3-manifolds in terms of the homology of $\Omega^\infty\,\mathrm{MTSO}(3)$. Corollary 1.0.5 shows that this is not the case.

Here is another consequence of Theorem 1.0.4:

Corollary 1.0.6. *Let $E \to B$ be an oriented fibre bundle over a closed oriented manifold with odd-dimensional closed fibres. Then* $\mathrm{sign}(E) = 0$.

This is an old theorem, which was first mentioned without proof by Atiyah [5] (perhaps the proof Atiyah had in mind is along the lines of the argument of the present paper). Proofs of 1.0.6 were given by Meyer [31] and Lück/Ranicki [27]. In fact, 1.0.4 and 1.0.6 are equivalent, as we will see in subsection 4.4.

There is a version of Theorem 1.0.4 for manifold bundles with boundary, such that the boundary is trivialized (section 7).

In dimensions of the form $4r + 1$, there is a real refinement of Theorem 1.0.3. More precisely, the odd signature operator has an index in real K-theory. This real index, however, is usually not zero. This is discussed in section 5

Theorem 1.0.3 is stronger than 1.0.4, because it also has consequences in mod p-cohomology. We prove two things for oriented 3-manifolds in that direction. Fix an oriented 3-manifold M. We will prove (in section 6):

- Fix $k \geq 1$. Then for almost all odd primes p, the map $\alpha^* : H^{4k-1}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-1}(B \operatorname{Diff}^+(M); \mathbb{F}_p)$ is zero (Theorem 6.0.4).
- Fix an odd prime p. Then $\alpha^* : H^{4k-1}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-1}(B \operatorname{Diff}^+(M); \mathbb{F}_p)$ is zero for an infinite number of values for k (Theorem 6.0.5).

In both cases, the primes to which the theorem applies does not depend on M.

In a companion paper [16] we show that all cohomology classes in $H^{*>0}(\mathrm{MTSO}(2m); \mathbb{Q})$ are detected on some bundle of $2m$-manifolds and that all classes in $H^{*>0}(\mathrm{MTSO}(2m+1); \mathbb{Q})$ which are not multiples of the Hirzebruch $\mathcal{L}$-class are detected on some $2m+1$-dimensional bundle. This means that Theorem 1.0.4 is the only vanishing theorem of this type.

1.1. **Outline of the paper.** Section 2 is a survey on the stable homotopy theory which is needed in this paper. We briefly discuss general Thom spectra, the Madsen-Tillmann-Weiss spectra, the Pontrjagin-Thom construction, the Madsen-Tillmann-Weiss map and Thom isomorphisms. Subsection 2.5 is devoted to a study of the component group $\pi_0(\mathrm{MTSO}(n))$. This is needed later in section 5. Section 3 provides the necessary constructions from index theory. In section 4, we discuss the odd signature operator and prove Theorem 1.0.3. Also, we show 1.0.4 and 1.0.5. Section 5 discusses the real index of the odd signature operator. Finally, in section 6, we discuss the vanishing theorem in finite characteristic. Section 7 discusses the extension of the results to the bounded case.

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2. Background material on Madsen-Tillmann-Weiss spectra

In this section, we review some material on the Madsen-Tillmann-Weiss spectra. Most of the results are standard except perhaps those concerning the component group of MTSO(n) in 2.5. For the subsections 2.1, 2.2 and 2.3, the reader is referred to the textbook [39] for proofs and much more details.

2.1. **Stable vector bundles and their Thom spectra.** For our purposes, a *stable vector bundle* V on a space X is a map $\xi_V : X \to \mathbb{Z} \times BO$. The *rank* of V is the locally constant function $X \stackrel{\xi_V}{\to} \mathbb{Z} \times BO \to \mathbb{Z}$. Given two ordinary vector bundles $V_i \to X$ of rank r_i, $i = 0, 1$, we can form their formal difference by the following procedure. Let $\mu : \mathbb{Z} \times BO \times \mathbb{Z} \times BO \to \mathbb{Z} \times BO$ be the Whitney sum map and $\iota : \mathbb{Z} \times BO \to \mathbb{Z} \times BO$ the inversion map. Furthermore, let $\xi_i : X \to \mathbb{Z} \times BO$ be classifying maps for V_i (composed with the inclusion $BO(r_i) \to BO$), then $V_0 - V_1$ is the stable vector bundle which is given by the composition

$$X \stackrel{\xi_0, \iota \circ \xi_1}{\to} \mathbb{Z} \times BO \times \mathbb{Z} \times BO \stackrel{\mu}{\to} \mathbb{Z} \times BO.$$

The rank of $V_0 - V_1$ is $r_0 - r_1$. Clearly, we can add and subtract stable vector bundles by means of the maps μ and ι. Furthermore, any ordinary vector bundle can be considered as a stable vector bundle.

The *Thom space* of a vector bundle $V \to X$ is the space $\mathrm{Th}(V) = X^V = \mathbb{D}(V)/\mathbb{S}(V)$, the quotient of the unit disc bundle by the unit sphere bundle. The *Thom spectrum* $\mathbb{T}\mathbf{h}(W)$ of a stable vector bundle W of rank d is produced as follows. Let $X_n := \xi_W^{-1}(\{d\} \times BO_{d+n})$; these subspaces form an exhaustive filtration $X_{-d} \subset X_{1-d} \subset \cdots \subset X$. Let $W_n := \xi_W^* L_{d+n}$ be the pullback of the $d+n$-dimensional universal vector bundle. Clearly, there is an isomorphism $W_{n+1}|_{X_n} \cong \mathbb{R} \oplus W_n$. The n^{th} space of $\mathbb{T}\mathbf{h}(W)$ is the Thom space $X_n^{W_n} := \mathbb{D}(W_n)/\mathbb{S}(W_n)$ of W_n and the structure maps are

$$\Sigma X_n^{W_n} \cong X_n^{\mathbb{R} \oplus W_n} \cong X_n^{W_{n+1}|_{X_n}} \hookrightarrow X_{n+1}^{W_{n+1}}.$$

The homotopy type of the spectrum $\mathbb{T}\mathbf{h}(W)$ depends only on the homotopy class of ξ_W. Furthermore, if W is an ordinary vector bundle, then the Thom spectrum is homotopy equivalent to the suspension spectrum $\Sigma^\infty X^W$ of the Thom space of W. In particular, the Thom spectrum of the trivial 0-dimensional bundle $\underline{0}$ on X is $\Sigma^\infty X_+$. Let W be a stable vector bundle and V an ordinary vector bundle. Given the description above, it is not hard to see that there is an inclusion map

$$\mathbb{T}\mathbf{h}(W) \to \mathbb{T}\mathbf{h}(W \oplus V).$$

Let $V \to X; W \to Y$ be two stable vector bundles. There is a canonical homotopy equivalence $\mathbb{T}\mathbf{h}(V) \wedge \mathbb{T}\mathbf{h}(W) \simeq \mathbb{T}\mathbf{h}(V \times W)$. If $X = Y$, we get a diagonal map $\mathrm{diag} : \mathbb{T}\mathbf{h}(V \oplus W) \to \mathbb{T}\mathbf{h}(V) \wedge \mathbb{T}\mathbf{h}(W)$. A special case is the diagonal $\mathbb{T}\mathbf{h}(V) \to \Sigma^\infty X_+ \wedge \mathbb{T}\mathbf{h}(V)$.

2.2. **Orientations and Thom isomorphisms.** Assume that A is an associative and commutative ring spectrum with unit (the rather old-fashioned notion of [3] is sufficient for our purposes). Let $V \to X$ be a stable vector bundle of rank $d \in \mathbb{Z}$. The cohomology $A^*(\mathbb{T}\mathbf{h}(V))$ is a graded left $A^*(X)$-module; a pair $(x, y) \in A^n(X) \times A^m(\mathbb{T}\mathbf{h}(V))$ is sent to the composition

$$x \cdot y : \mathbb{T}\mathbf{h}(V) \stackrel{\text{diag}}{\to} \Sigma^\infty X_+ \wedge \mathbb{T}\mathbf{h}(V) \stackrel{x \wedge y}{\to} \Sigma^n A \wedge \Sigma^m A \to \Sigma^{n+m} A.$$

A *Thom class* or *A-orientation* of V with A-coefficients is a cohomology class $v \in A^d(\mathbb{T}\mathbf{h}(V))$ such that for any $x \in X$, the image of v under the restriction map $A^d(\mathbb{T}\mathbf{h}(V)) \to A^d(\mathbb{T}\mathbf{h}(V_x)) \cong A^d(\mathbb{S}^d) \cong A^0(*)$ is a unit. This is equivalent to saying that $A^*(\mathbb{T}\mathbf{h}(V))$ is a free $A^*(X)$-module on the generator v or that he map $\mathrm{th}_V^A : A^*(X) \to A^{*+d}(\mathbb{T}\mathbf{h}(V))$; $x \mapsto x \cdot v$ is an isomorphism. If this is the case, then th_V^A is called the *Thom isomorphism.* If A is understood, then the superscript is often omitted.

More generally, we can define a *relative* Thom isomorphism. Let V be a stable vector bundle of rank d and let W be another stable vector bundle of rank e. Assume that V has a Thom class v. Let $\mathrm{th}^A_{W,W\oplus V} : A^*(\mathbb{T}\mathbf{h}(W)) \to A^{*+d}(\mathbb{T}\mathbf{h}(W \oplus V))$ be the homomorphism which maps $x \in A^n(\mathbb{T}\mathbf{h}(W))$ to the composition

$$\mathbb{T}\mathbf{h}(W \oplus V) \stackrel{\text{diag}}{\to} \mathbb{T}\mathbf{h}(W) \wedge \mathbb{T}\mathbf{h}(V) \stackrel{x \wedge v}{\to} \Sigma^n A \wedge \Sigma^d A \to \Sigma^{n+d} A;$$

this is an isomorphism of $A^*(X)$-modules. If $v \in A^d(\mathbb{T}\mathbf{h}(V))$ and $w \in A^e(\mathbb{T}\mathbf{h}(W))$ are Thom classes, then $\mathrm{th}^A_{W,W\oplus V}(v)$ is a Thom class for $V \oplus W$. If the Thom classes of different stable vector bundles are chosen compatibly in this way, then the Thom isomorphisms are compatible in the sense that $\mathrm{th}_{U\oplus V, U \oplus V \oplus W} \circ \mathrm{th}_{U,U\oplus V} = \mathrm{th}_{U,U\oplus V\oplus W}$ whenever U is an arbitrary stable vector bundle. We shall use the short notation $\mathrm{th}_V : A^*(\mathbb{T}\mathbf{h}(W)) \to A^*(\mathbb{T}\mathbf{h}(V \oplus W))$ if W is understood.

Examples: The examples of ring spectra which play a role in this paper are Eilenberg-Mac Lane spectra HR for commutative rings R as well as the complex K-theory spectrum K. It is well-known that a vector bundle which is oriented in the ordinary sense has a preferred[1] $H\mathbb{Z}$-Thom class and so it has a HR-Thom class for any ring R. A stable vector bundle has an $H\mathbb{Z}$-orientation if and only if $w_1(V) = 0$. Any complex vector bundle has a K-orientation and so does every complex stable vector bundle, i.e. a formal difference of complex vector bundles. However, there are several choices for these K-orientation. We follow the convention that the Thom class of a complex vector bundle $\pi : V \to X$ of rank n is represented by the complex

$$0 \to \pi^* \Lambda^0 V \stackrel{v\wedge}{\to} \pi^* \Lambda^1 V \stackrel{v\wedge}{\to} \pi^* \Lambda^2 V \to \ldots \pi^* \Lambda^n V \to 0.$$

The following observation is important for index theory. Let $V \to X$ be a real vector bundle. Then $V \otimes \mathbb{C}$ has a natural K-orientation. Therefore there is a relative Thom isomorphism

$$K^*(\mathbb{T}\mathbf{h}(-V)) \cong K^*(\mathbb{T}\mathbf{h}(-V \oplus V \otimes \mathbb{C})) \cong K^*(\mathbb{T}\mathbf{h}(V)). \tag{2.2.1}$$

[1]depending on the choice of a generator of $H_1(\mathbb{R}; \mathbb{R} \setminus 0; \mathbb{Z})$.

In this equation we used Bott periodicity to identify K^* with K^{*+2}. We will do this throughout the whole paper.

2.3. **The Pontrjagin-Thom construction.** Let M be a closed smooth oriented manifold of dimension n and let $\mathrm{Diff}^+(M)$ be the group of diffeomorphisms of M endowed with the Whitney C^∞-topology. We will study *smooth oriented M-bundles*, i.e. fibre bundles $f : E \to B$ with structural group $\mathrm{Diff}^+(M)$ and fibre M. Let $Q \to B$ be the associated $\mathrm{Diff}(M)$-principal bundle. The *vertical tangent bundle* is the oriented vector bundle $T_v E := Q \times_{\mathrm{Diff}^+(M)} TM \to Q \times_{\mathrm{Diff}^+(M)} M = E$. The *normal bundle* of f is the stable vector bundle $\nu(f) := -T_v E$.

If B is paracompact, then there is a fat embedding $j : E \to B \times \mathbb{R}^\infty$, i.e. $\mathrm{proj} \circ j = f$ and the image of j has a tubular neighborhood U. Moreover, the space of such fat embeddings is contractible. Collapsing everything outside U to the basepoint defines a map of spectra

$$\mathrm{PT}_f : \Sigma^\infty B_+ \to \mathbb{T}\mathbf{h}(\nu(f)),$$

the *Pontrjagin-Thom map* (or PT-map, for short). The map PT_f depends on a contractible space of choices. In particular, its homotopy class only depends on f. For more details on the PT-construction in this parameterized setting, see [19], section 3.

The Pontrjagin-Thom map can be used to define the *umkehr map* in generalized cohomology. Let $f : E \to B$ be a smooth fibre bundle of dimension n, A a ring spectrum and we assume that $\nu(f)$ has an A-orientation. The umkehr homomorphism $f_! : A^*(E) \to A^{*-n}(B)$ is defined as the composition

$$A^*(E) \stackrel{\mathrm{th}_{\nu(f)}}{\to} A^{*-n}(\mathbb{T}\mathbf{h}(\nu(f))) \stackrel{\mathrm{PT}_f^*}{\to} A^{*-n}(B). \tag{2.3.1}$$

The original application of the Pontrjagin-Thom construction was to give a bordism theoretic description of the homotopy groups of Thom spectra (or vice versa). Here is the most general version of this correspondence.

Theorem 2.3.2. *Let $V \to X$ be a stable vector bundle of rank $-n \in \mathbb{Z}$. If $-n > 0$, then $\pi_0(\mathbb{T}\mathbf{h}(V)) = 0$. If $n \geq 0$, then the group $\pi_0(\mathbb{T}\mathbf{h}(V))$ is isomorphic to the bordism group of triples (M^n, g, ϕ), where M^n is a closed smooth manifold, $g : M \to X$ a continuous map and $\phi : \nu(M) \cong g^*V$ a stable vector bundle isomorphism. Two triples (M_0, g_0, ϕ_0) and (M_1, g_1, ϕ_1) are bordant if there exists a bordism N from M_0 to M_1, a continuous map $h : N \to X$ such that $h|_{M_i} = g_i$ and a stable bundle isomorphism $\psi : \nu(N) \oplus \mathbb{R} \cong h^*V$ whose restriction to M_i is the isomorphism $\nu(N) \oplus \mathbb{R} \cong \nu(M_i) \stackrel{\phi_i}{\cong} g_i^* V$.*

Given a triple (M, g, ϕ), $c : M \to *$ the constant map, then the corresponding element in $\pi_0(\mathbb{T}\mathbf{h}(V))$ is the composition

$$\Sigma^\infty \mathbb{S}^0 \stackrel{\mathrm{PT}_c}{\to} \mathbb{T}\mathbf{h}(\nu(M)) \stackrel{g,\Phi}{\to} \mathbb{T}\mathbf{h}(V).$$

A detailed proof of this well-known result can be found in [39], ch. IV §7. Of course, this also gives an interpretation of the groups $\pi_k(\mathbb{T}\mathbf{h}(V)) \cong \pi_0(\mathbb{T}\mathbf{h}(V - \mathbb{R}^k))$.

2.4. **Madsen-Tillmann-Weiss spectra and Madsen-Weiss maps.** Let $n \geq 0$, let $BSO(n)$ be the classifying space for oriented Riemannian n-dimensional vector bundles and let $L_n \to BSO(n)$ be the universal oriented vector bundle. The reader should note that the space $BSO(0)$ is homotopy equivalent to the two-point space $\mathbb{S}^0$ and therefore it is *not* the classifying space for the group $SO(0)$. The most natural explanation for this phenomenon occurs in the framework of stacks. Let $\mathrm{Or}(\mathbb{R}^n)$ be the set of orientations of the vector space $\mathbb{R}^n$; the group $O(n)$ acts on $\mathrm{Or}(\mathbb{R}^n)$. The stack of oriented n-dimensional vector bundles is the quotient stack $\mathrm{Or}(\mathbb{R}^n)/\!\!/O(n)$. For $n \geq 1$, the $O(n)$-action on $\mathrm{Or}(\mathbb{R}^n)$ is transitive and hence $\mathrm{Or}(\mathbb{R}^n)/\!\!/O(n) \cong */\!\!/SO(n)$, while for $n = 0$, we have $\mathrm{Or}(\mathbb{R}^0)/\!\!/O(0) \cong \mathbb{S}^0$.

Definition 2.4.1. The Thom spectrum of the stable vector bundle $-L_n$ on $BSO(n)$ is called the *Madsen-Tillmann-Weiss spectrum* (or MTW-spectrum) and it is denoted by $\mathrm{MTSO}(n)$. Moreover, we denote by $\mathrm{MNSO}(n)$ the Thom spectrum of L_n.

Let $f : E \to B$ be a smooth oriented M-bundle. Recall that the space of orientation-preserving bundle maps $\lambda : T_v E \to L_n$ is contractible. Therefore the orientation defines a contractible space of maps $\kappa_E = \mathbb{T}\mathbf{h}(\lambda) : \mathbb{T}\mathbf{h}(-T_v E) \to \mathrm{MTSO}(n)$. The *Madsen-Tillmann-Weiss map* (or MTW-map) of the bundle $f : E \to B$ is the composition

$$\alpha_E := \kappa_E \circ \mathrm{PT}_f : \Sigma^\infty B_+ \to \mathrm{MTSO}(n), \tag{2.4.2}$$

which is defined uniquely up to a contractible space of choices.

For the universal oriented M-bundle $E_M \to B\,\mathrm{Diff}^+(M)$, we obtain a universal MTW-map

$$\alpha_{E_M} : \Sigma^\infty (B\,\mathrm{Diff}^+(M))_+ \to \mathrm{MTSO}(n).$$

On the other extreme, the constant map $M \to *$ is a smooth oriented M-bundle and its MTW-map is a map $\alpha_M : \Sigma^\infty \mathbb{S}^0 \to \mathrm{MTSO}(n)$.

By the Thom isomorphism,

$$H^*(BSO(n); R) = H^*(\Sigma^\infty BSO(n)_+; R) \cong H^{*-n}(\mathrm{MTSO}(n); R).$$

The cohomology of $BSO(n)$ is well-known. For example, if $\mathbb{F}$ is a field of characteristic different from 2, then

(2.4.3)
$$H^*(BSO(2m{+}1); \mathbb{F}) \cong \mathbb{F}[p_1, p_2, \ldots p_m]; \quad H^*(BSO(2m); \mathbb{F}) \cong \mathbb{F}[p_1, \ldots p_m, \chi]/(\chi^2 - p_m).$$

Let $f : E \to B$ be an oriented n-dimensional manifold bundle, let $\alpha_E : \Sigma^\infty B_+ \to \mathrm{MTSO}(n)$ be its MTW-map. An element $c \in H^*(BSO(n))$ can be interpreted as a characteristic class for oriented n-dimensional vector bundles and therefore we write $c(T_v E) \in H^*(E)$ for the pullback $\bar{\lambda}^* c$, where $\bar{\lambda} : E \to BSO(n)$ is any map underlying a bundle map $T_v E \to L_n$.

Proposition 2.4.4. *Let the notations be as above. Then*

$$\alpha_E^* \operatorname{th}_{-L_n}(c) = f_!(c(T_v(E))) \in H^{*-n}(B).$$

Proof. By definition

$$\alpha^* \operatorname{th}_{-L_n}(c) \overset{2.4.2}{=} \mathrm{PT}_f^* \, \mathbb{T}\mathbf{h}(\lambda)^* \operatorname{th}_{-L_n}(c) = \mathrm{PT}_f^* \operatorname{th}_{-T_v E}(c(T_v E)).$$

The second equality expresses the compatibility of Thom isomorphisms and pullbacks. ☐

Therefore any $c \in H^*(BSO(n))$ defines a characteristic class of oriented n-manifold bundles. We call these classes "generalized MMM-classes", because the case $n = 2$, $c = \chi^{i+1}$ gives the classes κ_i defined by Mumford [36], Miller [34] and Morita [35].

Recall the adjunction between the two functors Σ^∞ and Ω^∞: given a spectrum $\mathbf{E}$ and a space X, there is a natural bijection

$$[X, \Omega^\infty \mathbf{E}] \cong [\Sigma^\infty X_+, \mathbf{E}].$$

Under this adjunction, α_E corresponds to a map $B \to \Omega^\infty \mathrm{MTSO}(n)$, which is the original MTW-map studied in [29], [30], [20]. We will call this adjoint by the same name and denote it by the same symbol. There is no danger of confusion, because we keep our notation for spaces and spectra entirely disjoint. For the more computational purposes of the present paper, the spectra point of view is more transparent and convenient.

The adjoint $\Sigma^\infty(\Omega^\infty \mathbf{E})_+ \to \mathbf{E}$ of the identity on $\Omega^\infty \mathbf{E}$ induces a map

$$s : A^*(\mathbf{E}) \to A^*(\Omega^\infty \mathbf{E}),$$

the *cohomology suspension*, whenever A is a spectrum. If $A = H\mathbb{Q}$, then the right-hand-side is a graded-commutative $\mathbb{Q}$-algebra, but the left-hand-side is only a graded $\mathbb{Q}$-vector space. Let Λ denote the functor which associates to a graded module the free, graded-commutative algebra it generates; s extends to an algebra homomorphism

$$s : \Lambda(H^{*>0}(\mathbf{E}; \mathbb{Q})) \to H^*(\Omega_0^\infty \mathbf{E}; \mathbb{Q}). \tag{2.4.5}$$

This is an isomorphism by a classical result of algebraic topology, see [32], p. 262 f.

If X is a space and $\Sigma^\infty X_+ \to \mathbf{E}$ a map with adjoint $X \to \Omega^\infty \mathbf{E}$, then the following diagram commutes

$$\begin{tikzcd} A^*(\mathbf{E}) \arrow[r] \arrow[d, "s"] & A^*(\Sigma^\infty X_+) \arrow[d, equal] \\ A^*(\Omega^\infty \mathbf{E}) \arrow[r] & A^*(X). \end{tikzcd} \tag{2.4.6}$$

The rational cohomology of $\Omega^\infty \operatorname{MTSO}(n)$ can be easily computed using 2.4.3, 2.4.6 and 2.4.5.

2.5. **The component group of Madsen-Tillmann-Weiss spectra.** There are several maps which relate the spectra $\operatorname{MTSO}(n)$ for different values of n.

- The obvious bundle isomorphism $L_{n+1}|_{BSO(n)} \cong L_n \oplus \mathbb{R}$ induces a map of spectra $\eta : \operatorname{MTSO}(n) \to \Sigma \operatorname{MTSO}(n+1)$.
- The inclusion $-L_{n+1} \to \underline{0}$ of stable vector bundles on $BSO(n+1)$ yields a spectrum map $\omega : \operatorname{MTSO}(n+1) \to \Sigma^\infty BSO(n+1)_+$.
- The MTW-map of the oriented $\mathbb{S}^n$-bundle $BSO(n) \to BSO(n+1)$ is a map $\beta : \Sigma^\infty BSO(n+1)_+ \to \operatorname{MTSO}(n)$.

Proposition 2.5.1. *The maps η, ω and β form a cofibration sequence*

$$\operatorname{MTSO}(n+1) \xrightarrow{\omega} \Sigma^\infty BSO(n+1)_+ \xrightarrow{\beta} \operatorname{MTSO}(n) \xrightarrow{\eta} \Sigma \operatorname{MTSO}(n+1). \tag{2.5.2}$$

Proof. This follows immediately from Lemma 2.1 in [18]. □

The (homotopy) colimit of the sequence

$$\operatorname{MTSO}(0) \xrightarrow{\eta} \Sigma \operatorname{MTSO}(1) \xrightarrow{\eta} \Sigma^2 \operatorname{MTSO}(2) \to \dots$$

is the universal Thom spectrum $\widetilde{\mathrm{MSO}}$, the Thom spectrum of the universal 0-dimensional stable vector bundle $-L \to \mathrm{BSO}$ (which becomes $\mathbb{R}^n - L_n$ when restricted to $\mathrm{BSO}(n)$). The usual universal Thom spectrum MSO is the Thom spectrum of $L \to \mathrm{BSO}$. The spectra $\widetilde{\mathrm{MSO}}$ and MSO are homotopy equivalent: Let $\iota : \mathrm{BSO} \to \mathrm{BSO}$ be the inversion map, such that $\iota^* L = -L$. The map ι is covered by a bundle map $j : -L \to L$ which induces a homotopy equivalence $\mathbb{T}\mathbf{h}(j) : \widetilde{\mathrm{MSO}} \to \mathrm{MSO}$

The long exact homotopy sequence induced by 2.5.2 shows that the map η_* : $\pi_i(\operatorname{MTSO}(n)) \to \pi_i(\Sigma \operatorname{MTSO}(n+1))$ is an epimorphism if $i \leq 0$ and an isomorphism if $i < 0$. Therefore the inclusion $\Sigma^n \operatorname{MTSO}(n) \to \widetilde{\mathrm{MSO}}$ yields an isomorphism $\pi_i(\operatorname{MTSO}(n)) \cong \pi_{n+i}(\widetilde{\mathrm{MSO}}) \cong \pi_{n+i}(\mathrm{MSO}) \cong \Omega^{SO}_{n+i}$ (the oriented bordism group) for all $i < 0$. Therefore from 2.5.1, we get a commutative diagram; the rows are exact and the vertical maps are isomorphisms:

(2.5.3)

$$\begin{tikzcd}
\pi_0(\mathrm{MTSO}(n+1)) \arrow[r] \arrow[d] & \pi_0(\Sigma^\infty \mathrm{BSO}(n+1)_+) \arrow[r] \arrow[d] & \pi_0(\mathrm{MTSO}(n)) \arrow[r] \arrow[d] & \pi_{-1}(\mathrm{MTSO}(n+1)) \arrow[r] \arrow[d] & 0 \\
\pi_0(\mathrm{MTSO}(n+1)) \arrow[r] & \mathbb{Z} \arrow[r] & \pi_0(\mathrm{MTSO}(n)) \arrow[r] & \Omega_n^{SO} \arrow[r] & 0.
\end{tikzcd}$$

In order to study these groups further, we use the bordism-theoretic interpretation of $\pi_0(\mathrm{MTSO}(n))$ provided by Theorem 2.3.2.

It can be rephrased in such a way that $\pi_0(\mathrm{MTSO}(n))$ is the bordism group of oriented n-manifolds, where M_0 and M_1 are considered to be bordant if and only if there exists an oriented bordism N between them, an oriented n-dimensional vector bundle V on N and a stable bundle isomorphism $TN \cong V \oplus \mathbb{R}$. Clearly we can assume that N has no closed component and therefore, there is an actual isomorphism of vector bundles $TN \cong V \oplus \mathbb{R}$ by elementary obstruction theory. In other words, there is a nowhere vanishing tangential vector field on N which is the inward normal vector field on M_0 and the outward normal vector field on M_1. This bordism group is also known as Reinhardt's bordism group, see [38].

The maps in 2.5.3 have the following interpretation:

(1) $\pi_0(\mathrm{MTSO}(n+1)) \to \mathbb{Z}$ sends the bordism class of an oriented $n+1$-manifold M to its Euler number $\chi(M)$ (which is a cobordism invariant in Reinhardt's bordism).
(2) $\mathbb{Z} \to \pi_0(\mathrm{MTSO}(n))$ sends 1 to the bordism class of $\mathbb{S}^n$.
(3) $\pi_0(\mathrm{MTSO}(n)) \to \Omega_n^{SO}$ is the forgetful map.

Only the first claim needs a further justification. If $f : E \to B$ is an oriented $n+1$-manifold bundle, then the composition $\Sigma^\infty B_+ \overset{\alpha_E}{\to} \mathrm{MTSO}(n+1) \overset{\omega}{\to} \Sigma^\infty \mathrm{BSO}(n+1)_+$ is the composition of the Becker-Gottlieb transfer $\Sigma^\infty B_+ \to \Sigma^\infty E_+$ (see [13]) with the classifying map $\Sigma^\infty E_+ \to \Sigma^\infty \mathrm{BSO}(n)_+$ of $T_v E$. Therefore, if B is connected (and $n+1>0$), then the induced map on $\mathbb{Z} = \pi_0 \Sigma^\infty B_+ \to \Sigma^\infty BSO(n+1)_+ = \mathbb{Z}$ is the multiplication by the Euler number $\chi(M)$ of M, by Theorem 2.4 of [13].

Let $\mathrm{Eul}_n \subset \mathbb{Z}$ be the subgroup generated by all Euler numbers of oriented n-manifolds; the exact sequence 2.5.3 induces

$$0 \to \mathbb{Z}/\mathrm{Eul}_{n+1} \to \pi_0(\mathrm{MTSO}(n)) \to \Omega_n^{SO} \to 0. \tag{2.5.4}$$

The group $\mathrm{Eul}_n \subset \mathbb{Z}$ is easily computed. Its values are

$$\mathrm{Eul}_n = \begin{cases} 0; & n \not\equiv 0 \pmod 2; \\ 2\mathbb{Z}; & n \equiv 2 \pmod 4; \\ \mathbb{Z}; & n \equiv 0 \pmod 4. \end{cases}$$

The first case is clear. The third case follows from $\chi(\mathbb{S}^{4k}) = 2$ and $\chi(\mathbb{CP}^{2k}) = 2k+1$. The second case is implied by $\chi(\mathbb{S}^{4k+2}) = 2$ and the congruence $\chi(M^{4k+2}) \equiv 0 \pmod 2$ which follows from Poincaré duality in a straightforward manner.

The sequence 2.5.4 is always split, as we show now. If M is an oriented closed $(4m+1)$-dimensional manifold, then the *Kervaire semi-characteristic* $\mathrm{Kerv}(M) \in \mathbb{Z}/2$ is defined to be $\mathrm{Kerv}(M) := \sum_{i \geq 0} b_{2i}(M) = \sum_{i=0}^{2m} \dim b_i(M)$, where b_i is the real Betti number of M. By the proposition below, $\mathrm{Kerv}(M)$ defines a homomorphism $\pi_0(\mathrm{MTSO}(4m+1)) \to \mathbb{Z}/2$.

Proposition 2.5.5. *The Kervaire semi-characteristic of an oriented* $4m+1$*-manifold* M *only depends on its bordism class in* $\pi_0(\mathrm{MTSO}(4m+1))$.

Proof. It is enough to show the following: If N^{4m+2} is a connected oriented manifold with boundary M and if there is a nowhere vanishing vector field on N which is normal to the boundary, then $\mathrm{Kerv}(M) = 0$. Clearly, the double dN of N is closed and has a vector field without zeroes; thus $\chi(dN) = 0$ and therefore $\chi(N) = 0$. Let A be the image of $H^{2m+1}(M, N) \to H^{2m+1}(N)$.

Look at the long exact sequence of the pair (N, M) in real cohomology:

$$0 \to H^0(N, M) \to H^0(N) \to H^0(M) \to \ldots \to H^{2m}(M) \to H^{2m+1}(N, M) \to A \to 0.$$

We compute (in $\mathbb{Z}/2$)

$$\begin{aligned} 0 &= \sum_{i=0}^{2m+1} b_i(N; M) + \sum_{i=0}^{2m} b_i(N) + \sum_{i=0}^{2m} b_i(M) + \dim A = \\ &= \sum_{i=2m+1}^{4m+2} b_i(N) + \sum_{i=0}^{2m} b_i(N) + \sum_{i=0}^{2m} b_i(M) + \dim A = \\ &= \chi(N) + \mathrm{Kerv}(M) + \dim A = \mathrm{Kerv}(M) + \dim A. \end{aligned}$$

By Poincaré duality, the cup product pairing on A is skew-symmetric and nondegenerate, thus $\dim A \equiv 0 \pmod 2$. □

Let us summarize the description of the component group of $\mathrm{MTSO}(n)$.

(1) If $n \equiv 3 \pmod 4$, then $\pi_0(\mathrm{MTSO}(n)) \cong \Omega_n^{SO}$
(2) If $n \equiv 2 \pmod 4$, then the sequence 2.5.4 splits by $\pi_0(\mathrm{MTSO}(n)) \to \mathbb{Z}$; $[M] \mapsto \frac{1}{2}\chi(M)$.
(3) If $n \equiv 0 \pmod 4$, then 2.5.4 is also split. If M^{4m} is an oriented manifold, then $\mathrm{sign}(M) + \chi(M) \equiv 0 \pmod 2$ is immediate from the definition of the signature and the Euler number and from Poincaré duality. The map $\pi_0(\mathrm{MTSO}(n)) \to \mathbb{Z}$, $[M] \mapsto \frac{1}{2}(\mathrm{sign}(M) + \chi(M))$ is a splitting.
(4) If $n \equiv 1 \pmod 4$, then 2.5.4 is split by the Kervaire semi-characteristic $\pi_0(\mathrm{MTSO}(n)) \to \mathbb{Z}/2$.

3. Background material on index theory

In this section, we will present background material on index theory for bundles of compact manifolds. For details, the reader is referred to either the original source [10] or to the textbook [28].

There are two types of the K-theoretic index theorem: One for usual elliptic operators and another one for self-adjoint elliptic operators on a fibre bundle $f : E \to B$. In the former case, the index is an element in $K^0(B)$ while in the latter one we get an index in[2] $K^1(B)$.

Assume that $f : E \to B$ is a smooth fibre bundle on a paracompact space B with compact closed fibres. Assume that a fibrewise smooth Riemannian metric on the vertical tangent bundle T_vE is chosen. All vector bundles on E will be fibrewise smooth (i.e. the transition functions are smooth in the fibre-direction) and all hermitian metrics on vector bundles are understood to be smooth. All differential operations, like exterior derivatives and connections, will be fibrewise.

For an hermitian vector bundle $V \to E$, we denote $\Gamma_B(V) = \bigcup_{x\in B} \Gamma(E_x; V_x)$, where $E_x = f^{-1}(x)$ and $V_x = V|_{E_x}$. This family of vector spaces over B can be made into a vector bundle (of Fréchet spaces) by requiring that a section $s : B \to \Gamma_B(V)$ is continuous if the associated section of $V \to E$ is continuous in the C^∞-topology. Using the metrics on T_vE and V and a connection on V, we can define the L^2-Sobolev norms $\| \ldots \|_r$ on $\Gamma_B(V)$, for all $r \geq 0$. The completion with respect to this norm is a Hilbert bundle which we denote by $W^{2,r}_B(V)$.

There is a technical problem to overcome at this point; it is discussed and solved in [7], pp. 5, 13 f., 38-43. Namely, it is not quite true that the structural group of $W^{2,r}_B(V)$ is general linear group of an infinite-dimensional Hilbert space. The reason is that the action $\mathrm{Diff}(M) \curvearrowright W^{2,s}(M)$ is continuous only in the sense that $\mathrm{Diff}(M) \times W^{2,s}(M) \to W^{2,s}(M)$ is continuous, but not $\mathrm{Diff}(M) \to \mathrm{GL}(W^{2,s})$ when the latter has the norm topology. Instead, this map is continuous when $\mathrm{GL}(W^{2,s})$ has the compactly generated compact-open topology. Denote by $\mathrm{GL}(W^{2,s})_{co}$ the group with this topology. Then $\mathrm{GL}(W^{2,s})_{co}$ is contractible (this is much easier than Kuiper's theorem which asserts that $\mathrm{GL}(W^{2,s})$ is contractible). Moreover, $\mathrm{GL}(W^{2,s})_{co}$ acts continuously by conjugation on the space of Fredholm operators with a suitably redefined topology. This new space of Fredholm operators is homotopy equivalent to the original one.

Therefore the Hilbert bundles $W^{2,r}_B(V)$ are trivial and the trivialization is unique up to homotopy (in fact, the space of trivializations is contractible).

Let $V_0, V_1 \to E$ be two hermitian vector bundles and let $D : V_0 \to V_1$ be a vertical elliptic operator of order m. Then D has an extension to the bundle of Sobolev spaces $D : W^{2,s+m}_B(V) \to W^{2,s}_B(V)$, which consist of Fredholm operators. We choose, for any vector bundle V, an elliptic pseudodifferential operator A of order $-m/2$ which is invertible (for example $A_V = (1 + \nabla^*\nabla)^{-m/4}$ will do for any connection ∇ on V). The operator $A_{V_1} D A_{V_0}$ has order 0 and so it induces a family of

[2] We are using the Bott periodicity theorem without mentioning it. Therefore we identify K^1 with K^{-1}.

Fredholm operator $W^{2,0}_B(V_0) \to W^{2,0}_B(V_1)$. After an application of the trivializations above, we get a continuous map, denoted $\mathrm{ind}(D)$:

$$\mathrm{ind}(D) : B \to \mathrm{Fred}(H),$$

where H is a fixed separable, infinite-dimensional Hilbert space. The Atiyah-Jänich theorem states that the $\mathrm{Fred}(H)$ is a classifying space for complex K-theory and therefore we get an element $\mathrm{ind}(D) \in K^0(B)$. It does not depend on the choices involves.

On the other hand, if $D : \Gamma_B(V) \to \Gamma_B(V)$ is a formally self-adjoint elliptic operator of order $m \geq 0$, we get an index in $K^1(B)$. Here we consider the operator $A_V D A_V^*$, which is elliptic of order 0 and formally self-adjoint. It has the same kernel and the same positive and negative spectral spaces as the original D.

Thus we get a self-adjoint bounded Fredholm operator $D : W^{2,0}_B(V) \to W^{2,0}_B(V)$. In the same way as for ordinary elliptic operators, we get a map $B \to \mathrm{Fred}_{s.a.}(H)$, where $\mathrm{Fred}_{s.a.}(H)$ is the space of self-adjoint Fredholm operators on H endowed with the norm topology. Let $\mathrm{Fred}^{\pm}_{s.a.}(H) \subset \mathrm{Fred}_{s.a.}(H)$ be the subspace consisting operators A such that $\pm A$ is essentially positive (an operator is *essentially positive* if there exists an A-invariant subspace $U \subset H$ of finite codimension, such that $A|_U$ is positive definite). The two spaces $\mathrm{Fred}^{\pm}_{s.a}(H) \subset \mathrm{Fred}_{s.a.}(H)$ are open, closed and contractible (in fact, $\mathrm{Fred}^{\pm}_{s.a.}(H)$ is star-shaped with center $\pm\,\mathrm{id}$).

Let $\mathrm{Fred}^0_{s.a.} = \mathrm{Fred}_{s.a.}(H) \setminus (\mathrm{Fred}^+_{s.a.}(H) \cup \mathrm{Fred}^-_{s.a.}(H))$. Atiyah and Singer [12] showed that it has a very interesting topology: it has the homotopy type of the infinite unitary group $U(\infty)$. Thus it is a representing space for K^{-1}.

Returning to the self-adjoint family of operators D on $E \to B$, the map $B \to \mathrm{Fred}_{s.a.}(H)$ defines an element $\mathrm{ind}(D) \in K^1(B)$ (if D is essentially definite, this element is trivial).

3.1. **The topological index.** Let $f : E \to B$ be a smooth proper bundle, $\pi : T := T^*_v E \to E$ the vertical cotangent bundle and $\pi_0 : \mathbb{S}(T^*_v E) \to E$ its unit sphere bundle. Let $D : \Gamma_B(V_0) \to \Gamma_B(V_1)$ an elliptic differential operator. Recall that the *symbol* of D is a bundle map $\mathrm{smb}_D : \pi^* V_0 \to \pi^* V_1$ which is an isomorphism outside the zero section (this is the definition of ellipticity). If D has order 1, then the symbol is

$$\mathrm{smb}_D(\xi)v = i(D(fs) - fDs),$$

where ξ is a vertical cotangent vector at $x \in E$, f is a smooth function such that $df_x = \xi$ and s is a section of V_0 such that $s(x) = v$. For higher orders, there is a more complicated formula, which we will not need here.

We will constantly identify the vertical cotangent and the vertical tangent bundle. The symbol smb_D defines the *symbol class* $[\mathrm{smb}_D]_0 \in K^0(T; T \setminus 0) = K^0(E^T)$ of D.

Following [6], we can associate a symbol class $[\mathrm{smb}_D]_1 \in K^{-1}(E^T)$ to a self-adjoint elliptic operator D. Consider the symbol $\mathrm{smb}_D : \pi^* V \to \pi^* V$. It is a self-adjoint endomorphism of $\pi^* V$ and it is an isomorphism away from the zero section. Let $\tilde{\pi} : T \oplus \mathbb{R} \to E$. We define $[\mathrm{smb}_D]_1$ to be the class in $K^{-1}(E^T) = K^0((T, T \setminus 0) \times (\mathbb{R}, \mathbb{R} \setminus 0))$ represented by the complex

$$0 \to \tilde{\pi}^* V \overset{\tilde{\mathrm{smb}}_D}{\to} \tilde{\pi}^* V \to 0,$$

where $\tilde{\mathrm{smb}}_D$ is given at the point $(x,t) \in T \oplus \mathbb{R}$ by $\tilde{\mathrm{smb}}_{D(x,t)} := (\mathrm{smb}_D)_x - it\,\mathbf{1}$. Actually, [6] give a different formula, but the passage between the two formulations is by an elementary deformation. We leave it to the reader to figure that out.

Recall the relative Thom isomorphism 2.2.1 $\mathrm{th}_{-T_vE\otimes\mathbb{C}} : K^*(E^{T_vE}) \to K^*(E^{-T_vE})$. The Atiyah-Singer family index theorem ([10] for the usual case, [6] for the self-adjoint case) states that in both cases ($i = 0, 1$)

$$\mathrm{ind}(D) = \beta^{-d}\,\mathrm{PT}_f^*\,\mathrm{th}_{-T_vE\otimes\mathbb{C}}([\mathrm{smb}_D]_i) \in K^i(B). \tag{3.1.1}$$

3.2. **Universal operators.** Now we assume that the first order elliptic operator D on the n-dimensional oriented bundle $E \to B$ family is *universal on the symbolic level.* By that expression, we mean that there exist $SO(n)$-representations W_0 and W_1 and an $SO(n-1)$-equivariant isomorphism $\gamma : W_0 \to W_1$ such that

1. as Hermitian vector bundles, V_0 and V_1 are isomorphic to the associated bundle $\mathrm{Fr}_v(E) \times_{SO(n)} W_i \to E$;
2. the symbol smb_D restricted to the unit cotangent sphere bundle equals the bundle map $\mathrm{Fr}_v(E) \times_{SO(n-1)} W_0 \to \mathrm{Fr}_v(E) \times_{SO(n-1)} W_1$ induced by γ.

The trivial vector bundles $\mathbb{R}^n \times W_i$ on $\mathbb{R}^n$ are $SO(n)$-equivariant and the map γ defines an $SO(n)$-equivariant isomorphism $\mathbb{S}^{n-1} \times W_0 \to \mathbb{S}^{n-1} \times W_0$. Therefore, (W_0, W_1, γ) defines a class $\sigma_D \in K^0_{SO(n)}(\mathbb{D}^n, \mathbb{S}^{n-1})$. The image of σ_D under the standard homomorphism $K^0_{SO(n)}(\mathbb{D}^n, \mathbb{S}^{n-1}) \to K^0(\mathbb{D}(L_n), \mathbb{S}(L_n)) \cong K^0(\mathrm{MNSO}(n))$ is denoted by the same symbol. Clearly, the class σ_D pulls back to the symbol class $[\mathrm{smb}_D]$ under the map $\mathbb{T}\mathbf{h}(T_vE) \to \mathrm{MNSO}(n)$. By the Thom isomorphism $K^0(\mathrm{MNSO}(n)) \cong K^0(\mathrm{MTSO}(n))$, we get a class $\mathrm{th}\,\sigma_D \in K^0(\mathrm{MTSO}(n))$.

The index theorem for the symbolically universal operator D on the fibre bundle $f : E \to B$ reads:

$$\mathrm{ind}(D) = \alpha_E^*\,\mathrm{th}\,\sigma_D. \tag{3.2.1}$$

Similarly, if $V_0 = V_1 = V$ and γ is self-adjoint, we get a class $\sigma_D \in K^1(\mathrm{MNSO}(n))$ and $\mathrm{th}\,\sigma \in K^1(\mathrm{MTSO}(n))$ and the index theorem is expressed by the same formula as in 3.2.1.

4. The index theorem for the odd signature operator

4.1. **The signature operators.** Let M be a closed oriented Riemannian manifold of dimension n. Recall that there is the Hodge star operator[3] $* : \mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M)$. The star operator is an complex-linear isometry and satisfies $** =$

[3] There might exist different sign conventions about $*$. We are constantly using the definition given in [9].

$(-1)^{k(n-k)} : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$. The adjoint $d^{ad} : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ of the exterior derivative can be written as $d^{ad} = (-1)^{n(k+1)+1} * d*$.

If $n = 2m$, then one introduces the involution $\tau := i^{k(k-1)+m}*$ on k-forms [9], p. 574. Then $D_{2m} = d + d^{ad}$ satisfies $D_{2m}\tau = -\tau D_{2m}$. If $\mathcal{A}^*_{\pm}(M)$ denote the ± 1-eigenbundles of τ, then the operator $D : \mathcal{A}^*_+(M) \to \mathcal{A}^*_-(M)$ is the (even) *signature operator*. This is an elliptic differential operator of order 1 whose index is the same as the signature of M.

Following [6], we introduce the odd signature operator on a $2m-1$-dimensional closed oriented Riemannian manifold M. Note that $** = 1$ and $d^{ad} = (-1)^k * d* : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$. The *odd signature operator* $D = D_{2m-1} : \bigoplus_{p \geq 0} \mathcal{A}^{2p}(M) \to \bigoplus_{p \geq 0} \mathcal{A}^{2p}(M)$ is defined to be

$$D_{2m-1}\phi = i^m(-1)^{p+1}(*d - d*)\phi$$

whenever $\phi \in \mathcal{A}^{2p}(M)$.

A straightforward, but tedious, calculation shows that

(1) D is formally self-adjoint and elliptic.
(2) $D^2 = \Delta = (d + d^{ad})^2$, the Laplace-Beltrami operator.

Moreover, one observes that

$$\ker(D) = \ker(\Delta) = \bigoplus_{p \geq 0} H^{2p}(M; \mathbb{C}) \tag{4.1.1}$$

and that consequently

$$\dim \ker D = \sum_{p \geq 0} \dim H^{2p}(M; \mathbb{C}), \tag{4.1.2}$$

which is the main property needed for the proof of Theorem 1.0.3.

Both signature operators are symbolically universal. The even one is associated with the $SO(2m-1)$-equivariant isomorphism of $SO(2m)$-representations

$$i(\epsilon - *\epsilon*) : \Lambda^*_+(\mathbb{R}^{2m}) \otimes \mathbb{C} \to \Lambda^*_-(\mathbb{R}^{2m}) \otimes \mathbb{C},$$

where ϵ denotes wedge multiplication with the last standard basis vector.

The odd signature operator is associated with the representation $\Lambda^{ev}(\mathbb{R}^{2m-1})$ and the endomorphism

$$i^{m-1}(-1)^p(*\epsilon - \epsilon*). \tag{4.1.3}$$

Abbreviate $\mathbb{R}^n_0 := \mathbb{R}^n \setminus 0$. Let $\sigma_{2m-1} \in K^{-1}_{SO(2m-1)}(\mathbb{R}^{2m-1}, \mathbb{R}^{2m-1}_0)$ be the universal symbol class of the odd signature operator and $\sigma_{2m} \in K^0_{SO(2m)}(\mathbb{R}^{2m}, \mathbb{R}^{2m}_0)$ be the universal symbol class of the even signature operator. We denote their images in $K^*(\mathrm{MNSO}(n))$ by the same symbol.

4.2. **The vanishing theorem.** Let $f : E \to B$ be a smooth oriented M-bundle, M a closed oriented $(2m-1)$-manifold. Assume that we choose a Riemannian metric on the vertical tangent bundle (for any bundle on a paracompact base space such a metric exists; the space of these metrics is contractible). The odd signature operators on the fibres of f fit together to a family of self-adjoint elliptic differential operators. Therefore we have the family index

$$\mathrm{ind}(D) \in K^1(B),$$

which does not depend on the auxiliary Riemannian metric, but which is an invariant of smooth oriented M-bundles. In the universal case, we get an element $\mathrm{ind}(D) \in K^1(B\,\mathrm{Diff}^+(M))$.

The proof of Theorem 1.0.3 is an immediate consequence of 4.1.2 and Theorem 4.2.1 below. Theorem 4.2.1 is well-known to some people working in operator theory, see e.g. [14], 5.1.4. and it is certainly implicitly contained in [12]. I have included the following rather elementary proof for the convenience of the reader.

Theorem 4.2.1. *Let B be a space and let $A : B \to \mathrm{Fred}^0_{s.a}(H)$, $x \mapsto A_x$ be a continuous map such that $x \mapsto \dim\ker A_x$ is locally constant. Then A is homotopic to a constant map.*

Proof. Step 1: First we show that we can deform A into a family A' consisting of invertible operators.

To this end, we note that because the dimension of $\ker A_x$ is locally constant, the union $\ker(A) := \bigcup_{x\in B}\ker(A_x)$ is a (finite-dimensional) vector bundle on B. Therefore the projection operator p_x onto the kernel of A_x depends continuously on x and p_x commutes with A_x because A_x is self-adjoint. Therefore $A_x + tp_x$ is Fredholm for all $t \in \mathbb{R}$ and $\mathrm{Spec}(A_x + tp_x) = \mathrm{Spec}\,A_x \setminus \{0\} \cup \{t\} \subset \mathbb{R}_{\neq 0}$. Thus for $t \neq 0$, $A_x + tp_x$ is invertible (and neither essentially negative nor positive).

Step 2: By step 1, we assume that A_x is invertible for all $x \in B$, in other words $\mathrm{Spec}(A_x) \subset \mathbb{R}\setminus 0$ for all $x \in B$. Let $h : \mathbb{R}\setminus 0 \to \mathbb{R}$ be the signum function. For any x and any $t \in [0,1]$, the operator

$$th(A) + (1-t)A$$

is a self-adjoint invertible operator (the latter statement is easy to see because A and $h(A)$ commute). For $t = 0$, we get A and for $t = 1$, we get $h(A)$ which is a self-adjoint involution which is neither essentially positive nor negative.

Step 3: By step 2, we can assume that A is a map from B into the space $\mathcal{P}(H)$ of all involutions F on H such that $\mathrm{Eig}(F, \pm 1)$ are both infinite-dimensional. Let us show that $\mathcal{P}(H)$ is contractible. The unitary group $U(H)$ acts transitively on $\mathcal{P}(H)$ (by conjugation) and the isotropy group at a given F_0 is $U(\mathrm{Eig}(F_1, 1)) \times U(\mathrm{Eig}(F_0, -1))$. Thus we have a continuous bijection

$$U(H)/U(\mathrm{Eig}(F_0, 1)) \times U(\mathrm{Eig}(F_0, -1)) \to \mathcal{P}.$$

The map $U(H) \to \mathcal{P}(H)$, $u \mapsto uF_0u^{-1}$ has a local section[4] and thus the bijection above is a homeomorphism. The left hand side space is contractible by Kuiper's

[4]Here is a construction of the local section. Let $H_\pm := \mathrm{Eig}(F_0; \pm 1)$. For a given F, let u_F be $\frac{1}{2}(1 \pm F)$ on $H_\pm$. The operator u_F depends continuously on F; $u_{F_0} = 1$. Therefore, for F close to F_0, u_F is isomorphism. An application of the Gram-Schmidt process defines a continuous family

theorem [26] and the long exact homotopy sequence, which completes the proof of the theorem. □

In the proof of the theorem we had the choice between two different contractible spaces of nullhomotopies of A; in the first step, we could choose either a positive value or a negative value of the real parameter t (put in another way: the spectral value can be pushed either in the positive or in the negative direction). The concatenation of these two nullhomotopies defines a map $B \to \Omega \operatorname{Fred}^0_{s.a.} \simeq \Omega U \simeq \mathbb{Z} \times BU$, in other words an element in $K^0(B)$. It is not hard to see that this is the same as the class of the bundle $\ker(A) \to B$.

In the case of the odd signature operator on the smooth oriented fibre bundle, this is the K-theory class of the flat bundle $\bigoplus_{p\geq 0} H^{2p}(E/B;\mathbb{C})$ of even cohomology groups. This K-theory class is a characteristic class of smooth oriented fibre bundle, nevertheless, it is *not* induced by an element in $K^0(\mathrm{MTSO}(2m-1))$. This can be seen as follows. There exist odd-dimensional manifold bundles $f: E \to B$ such that $\sum_{p\geq 0}[H^{2p}(E/B;\mathbb{C})] \neq 0 \in K^0(B)$. For example, ones takes orientation reversing involutions on $\mathbb{S}^1$ and $\mathbb{S}^2$. The diagonal action $\mathbb{Z}/2 \curvearrowright \mathbb{S}^1 \times \mathbb{S}^2$ is then orientation-preserving. The bundle $E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (\mathbb{S}^1 \times \mathbb{S}^2) \to B\mathbb{Z}/2)$ has the desired property. On the other hand $K^0_{SO(2m-1)}(\mathbb{R}^{2m-1}, \mathbb{R}^{2m-1}_0) \cong K^{1+\tau}_{SO(2m-1)}(*)$ by the Thom isomorphism in twisted K-theory ([17]). Here τ is the twist induced from the central extension $Spin^c(2m-1) \to SO(2m-1)$. On the other hand, $K^{1+\tau}_{SO(2m-1)} = 0$, see [17], p.11. By the Atiyah-Segal completion theorem [8], it follows that $K^0(\mathrm{MTSO}(2m-1)) \cong K^0(\mathrm{MNSO}(2m-1)) = (K^0_{SO(2m-1)}(\mathbb{R}^{2m-1}, \mathbb{R}^{2m-1}_0))^\wedge = 0$.

4.3. **Cohomology calculation.** In this section, we indicate how Theorem 1.0.4 is derived from Theorem 1.0.3. The computation is at least implicitly done in [9] and [6] and we shall give only a sketch. First note that the second statement of Theorem 1.0.4 is an immediate consequence of the first one in view of 2.4.4. The Atiyah-Singer index theorem 3.2.1 implies that Theorem 1.0.3 is equivalent to the following result; this formulation is what we actually need in the sequel.

Theorem 4.3.1. *Let $\sigma_{2m-1} \in K^1(\mathrm{MNSO}(2m-1))$ be the universal symbol class of the signature operator. Then for any smooth oriented bundle $E \to B$ of $(2m-1)$-dimensional closed manifolds, we have $\alpha_E^* \operatorname{th}(\sigma_{2m-1}) = 0$.*

Consider the following commutative diagram

$$\begin{tikzcd} K^0(\mathrm{MNSO}(2m)) \arrow[r, "\rho^*"] \arrow[d, "\operatorname{th}_{-L_{2m}\otimes\mathbb{C}}"] & K^0(\Sigma^1 \mathrm{MNSO}(2m-1)) \arrow[d, "\operatorname{th}_{-L_{2m-1}\otimes\mathbb{C}}"] \\ K^0(\mathrm{MTSO}(2m) \arrow[r, "\eta^*"] \arrow[d, "\mathrm{ch}"] & K^0(\Sigma^{-1} \mathrm{MTSO}(2m-1)) \arrow[d, "\mathrm{ch}"] \\ H^*(\mathrm{MTSO}(2m);\mathbb{Q}) \arrow[r, "\eta^*"] & H^*(\mathrm{MTSO}(2m-1);\mathbb{Q}). \end{tikzcd} \tag{4.3.2}$$

$F \mapsto u_F$ of unitary operators on a neighborhood of F_0 such that $u_F(H_\pm) = \operatorname{Eig}(F;\pm 1)$, in other words, $u_F F u_F^{-1} = F_0$, which is what we want.

Let $\tilde{\mathcal{L}} \in H^*(BSO;\mathbb{Q})$ be the multiplicative sequence in the Pontrjagin classes associated with the formal power series $\sqrt{x}\coth(\frac{\sqrt{x}}{2})$. Recall that the Hirzebruch $\mathcal{L}$-class is associated with with the formal power series $\sqrt{x}\coth(\sqrt{x})$. Note that the degree $4k$ parts in $H^*(BSO(2m))$ are related by

$$\tilde{\mathcal{L}}_{4k} = 2^{m-k}\mathcal{L}_{4k} \in H^{4k}(BSO(2m);\mathbb{Q}). \tag{4.3.3}$$

Proposition 4.3.4. (1) *The image of* $\sigma_{2m} \in K^0(\mathrm{MTSO}(2m))$ *under the restriction homomorphism* $\rho^* : K^0(\mathrm{MTSO}(2m)) \to K^0(\Sigma\,\mathrm{MTSO}(2m-1)) = K^1(\mathrm{MTSO}(2m-1))$ *coincides with* $2\sigma_{2m-1}$.

(2) *The image of* σ_{2m} *under* $\mathrm{ch}\circ\mathrm{th}_{-L_{2m}\otimes\mathbb{C}}$ *in* $H^*(\mathrm{MTSO}(2m);\mathbb{Q})$ *is the class* $\tilde{\mathcal{L}}$.

Proof. The first part is contained in the proof of Lemma 4.2 in [6]. The second part is done in [9], section 6. We leave it to the reader to translate the proofs into the present more abstract notation. □

Theorem 1.0.4 follows immediately from 4.3.1, 4.3.3, 4.3.4.

4.4. **Applications of Theorem 1.0.4.**

Proof of Corollary 1.0.5: Recall the power series expansion

$$\sqrt{x}\coth(\sqrt{x}) = \sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}x^{2k}$$

and recall that B_{2k} is a nonzero rational number. On the other hand $H^*(BSO(3)) = \mathbb{Q}[p_1]$ and therefore $\mathcal{L} = \sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}p_1^k$. Thus the components of $\mathcal{L}$ form an additive basis of $H^*(BSO(3))$. Therefore, by Theorem 1.0.4, $H^*(\mathrm{MTSO}(3);\mathbb{Q}) \to H^*(\Sigma^\infty(B\,\mathrm{Diff}^+(M))_+;\mathbb{Q})$ is trivial. By 2.4.5, this finishes the proof. □

Proof of Corollary 1.0.6. Because $TE \cong f^*TB \oplus T_vE$, we have

$$\mathrm{sign}(E) = \langle \mathcal{L}(TE);[E]\rangle = \langle \mathcal{L}(f^*TB)\mathcal{L}(T_vE);[E]\rangle = \langle \mathcal{L}(TB)f_!(\mathcal{L}(T_vE));[B]\rangle. \tag{4.4.1}$$

By Theorem 1.0.4, $f_!(\mathcal{L}(T_vE)) = 0$. □

To derive 1.0.4 from 1.0.6, observe first that $H_*(B\,\mathrm{Diff}^+(M);\mathbb{Q}) \cong \Omega_*^{fr}(B\,\mathrm{Diff}^+(M))\otimes\mathbb{Q}$ (the framed bordism group) by Pontrjagin's theorem and Serre's finiteness theorem. Therefore, to show that $\alpha_M^*\,\mathrm{th}\,\mathcal{L} = 0$, it suffices to show that $h^*\alpha_M^*\,\mathrm{th}\,\mathcal{L} = 0$ whenever $h : B \to B\,\mathrm{Diff}^+(M)$ is a map with B a framed manifold, classifying an M-bundle $f : E \to B$. If B is framed, then $\mathcal{L}(TB) = 1$ and therefore by 4.4.1 and 1.0.6

$$0 = \mathrm{sign}(E) = \langle \mathcal{L}(TE);[E]\rangle = \langle f_!(\mathcal{L}(T_vE))\mathcal{L}(TB);[B]\rangle = \langle f_!(\mathcal{L}(T_vE));[B]\rangle. \tag{4.4.2}$$

Therefore 1.0.4 follows.

5. A real refinement and the one-dimensional case

Let $f : E \to B$ be a smooth oriented fibre bundle of fibre dimension $2m-1$. Recall the formula for the odd signature operator: $D = i^m(-1)^{p+1}(*d - d*)$ on a $(2m-1)$-dimensional manifold. If m is odd ($2m-1 = 1, 5, \ldots$), then $-iD$ is a *real, skew-adjoint* operator, acting on real-valued differential forms. As such, it has an index in $KO^{-1}(B)$, compare [11].

The question we consider is whether this refined index is also trivial. We have to check whether the argument in the proof of Theorem 4.2.1 goes through with $-iD$ instead of D in the space of real, skew-adjoint Fredholm operators. It turns out that step 2 can be changed appropriately (we deform an invertible operator into one with $F^2 = -1$). The argument for step 3 can be applied to the space of skew-adjoint real Fredholm operators F with $F^2 = -1$, because Kuipers theorem is true for the isometry group of a real Hilbert space as well. The problem is with step 1.

Let $H \cong \bigoplus_{p \geq 0} H^{2p}(E/B; \mathbb{R}) \to B$ be the finite-dimensional real vector bundle formed out of the kernels of the real odd signature operator.

In order to make sense out of the deformation in step 1, it is not enough to know that H is a real vector bundle, but also that H admits a skew-adjoint invertible endomorphism. Such an endomorphism is, up to homotopy, the same as a complex structure on H. Therefore:

Theorem 5.0.3. *Let $f : E \to B$ be an oriented smooth M-bundle, M of dimension $4r+1$. Then the real family index of the odd signature operator $\operatorname{ind}_{\mathbb{R}} D \in KO^{-1}(B)$ is trivial if and only if the the K-theory class $[H] \in KO^0(B)$ lies in the image of the realification map $K^0(B) \to KO^0(B)$.*

The first obstruction to find a complex structure on H is of course the parity of its dimension $\dim H \pmod 2$. This agrees with the Kervaire semi-characteristic $\mathrm{Kerv}(M)$. More generally, we can interpret this result in terms of the exact sequence

$$K^{-2}(B) \xrightarrow{\gamma} KO^0(B) \xrightarrow{\delta} KO^{-1}(B),$$

compare e.g. [24], Thm 5.18. The map γ is the inverse to the Bott map, composed with the realification map $K^0 \to KO^0$ and δ is the product with the generator of $KO^{-1}(*) \cong \mathbb{Z}/2$. The image $\delta([H]) \in KO^{-1}(B)$ agrees with the real index. So the real index vanishes if and only if there is a complex structure on H.

It is worth to study the 1-dimensional case explicitly. The MTW-spectrum is $\mathrm{MTSO}(1) \cong \Sigma^{-1}\Sigma^\infty \mathbb{S}^0$. It is well-known that $\mathrm{Diff}^+(\mathbb{S}^1) \simeq \mathbb{S}^1$; therefore $B\,\mathrm{Diff}^+(\mathbb{S}^1) \simeq \mathbb{CP}^\infty$. The MTW-map $\alpha : \Sigma^\infty \mathbb{CP}^\infty_+ \to \Sigma^{-1}\Sigma^\infty\mathbb{S}^0$ can be identified with the *circle transfer*. The restriction $\Sigma^\infty\mathbb{S}^0 \to \Sigma^{-1}\Sigma^\infty\mathbb{S}^0$ of α to the basepoint is simply the generator $\eta \in \pi_1(\Sigma^\infty\mathbb{S}^0) \cong \mathbb{Z}/2$.

The odd signature operator on $\mathbb{S}^1$ is simply $D = -i * d$ on $C^\infty(\mathbb{S}^1)$. If $\mathbb{S}^1$ has a Riemannian metric with volume a and x is a coordinate $\mathbb{S}^1 \to \mathbb{R}/a\mathbb{Z}$ preserving orientation and length, then $D = -i\frac{d}{dx}$. The symbol is $\mathrm{smb}_D(dx) = -1$. Using this, it is easy to see that $\sigma_1 \in K^{-1}_{SO(1)}(\mathbb{R}, \mathbb{R}_0) = K^0(\mathbb{R}^2, \mathbb{R}^2_0)$ is the Bott class. Thus the universal symbol is a generator of $K^{-1}(\mathrm{MTSO}(1)) \cong \mathbb{Z}$.

The vanishing theorem 1.0.3 in this case can be obtained much easier because $K^{-1}(\mathbb{CP}^\infty) = 0$. In fact, the vanishing theorem for the topological index follows immediately from this fact, without any use of elliptic operator theory.

On the other hand, the Kervaire semi-characteristic of $\mathbb{S}^1$ is clearly nonzero and therefore the real index of the signature operator is nonzero; it is a generator of $KO^{-1}(\mathbb{CP}^\infty) \cong \mathbb{Z}/2$ (the latter isomorphism follows easily from the main result of [4]). The restriction of the real index to the basepoint is the generator of $KO^{-1}(*) = \mathbb{Z}/2$.

6. Vanishing theorems in mod p cohomology and an open problem

We have seen that for any oriented closed 3-manifold M, the map $\alpha_{E_M} : B\operatorname{Diff}^+(M) \to \mathrm{MTSO}(3)$ is trivial in rational cohomology. What we do not know is whether there exists an oriented closed 3-manifold M such that $\alpha_{E_M} : B\operatorname{Diff}^+(M) \to \Omega^\infty \mathrm{MTSO}(3)$ is nontrivial in homology.

In this section, we sketch two methods to derive from 1.0.3 that $\alpha_{E_M}^* : H^{4k-3}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-3}(B\operatorname{Diff}^+(M); \mathbb{F}_p)$ vanishes for certain values of k and primes p.

Theorem 6.0.4. *For any oriented closed 3-manifold M and for any $k \geq 1$, the map $\alpha_{E_M}^* : H^{4k-3}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k-3}(B\operatorname{Diff}^+(M); \mathbb{F}_p)$ is trivial for all primes p with $p \geq 2k$ and p not dividing the numerator of B_k (these are almost all primes, for a fixed k).*

Theorem 6.0.5. *For any oriented closed 3-manifold M and for any odd prime p, the map $\alpha_{E_M}^* : H^{4k+1}(\mathrm{MTSO}(3); \mathbb{F}_p) \to H^{4k+1}(B\operatorname{Diff}^+(M); \mathbb{F}_p)$ is trivial when $k = \frac{1}{2}(p-1)i$ for some $i \in \mathbb{N}$.*

Note that neither of the sets of pairs (k, p) provided by the two theorems contains the other one. Also, they do not exhaust all values of (k, p). Neither theorem makes a statement about the prime 2. The methods of the proof of both theorems can be used to derive vanishing theorems for all odd dimensions (and the method of 6.0.5 gives a result about the prime 2 as well), but here we confine ourselves to the case of dimension 3.

Proof of Theorem 6.0.4: The symbol of the odd signature operator $\mathrm{th}(\sigma) \in K^{-1}(\mathrm{MTSO}(3))$, when considered as a map $\mathrm{MTSO}(3) \to \Sigma^{-1}K$, can be lifted to connective K-theory, i.e. to a map

$$\kappa : \mathrm{MTSO}(3) \to \Sigma^{-3}\mathbf{k}.$$

The composition $\kappa \circ \alpha : B\operatorname{Diff}^+(M)_+ \to \mathrm{MTSO}(3) \to \Sigma^{-3}\mathbf{k}$ is still nullhomotopic. The theorem follows from a theorem of Adams [2], [1] about the spectrum cohomology of $\mathbf{k}$. In general, the class $s_r := r!\,\mathrm{ch}_r \in H^{2r}(BU; \mathbb{Z})$ is not a spectrum cohomology class, i.e. it does not lie in the image of the cohomology suspension $H^{2r}(\mathbf{k}; \mathbb{Z}) \to H^{2r}(BU; \mathbb{Z})$. The result of [2], [1] is that a certain multiple $m(r)\,\mathrm{ch}_r$ actually *is* a spectrum cohomology class. The number $m(r)$ is given by $m(r) := \prod_p p^{[\frac{r}{p-1}]}$. The product goes over all prime numbers and for $x \in \mathbb{R}$, $[x]$ is

the largest integer which is less or equal than x (thus, it involves only primes p with $p-1 \leq r$). Moreover, $u_r := m(r)\,\mathrm{ch}_r$ is a generator of $H^{2r}(\mathbf{k};\mathbb{Z}) \cong \mathbb{Z}$, $r \geq 0$ (all other cohomology groups of $\mathbf{k}$ are trivial). If p is an odd prime then $H^*(\mathrm{MTSO}(3);\mathbb{Z})$ has no p-torsion and so $H^*(\mathrm{MTSO}(3);\mathbb{Z}) \otimes \mathbb{F}_p \cong H^*(\mathrm{MTSO}(3);\mathbb{F}_p)$. Therefore, if $\kappa^*(\Sigma^{-3}u_r) \in H^{2r-3}(\mathrm{MTSO}(3))$ reduces to a generator of $H^{2r-3}(\mathrm{MTSO}(3);\mathbb{F}_p)$, then $\alpha : H^{2r-3}(\mathrm{MTSO}(3);\mathbb{F}_p) \to H^{2r-3}(B\,\mathrm{Diff}^+(M);\mathbb{F}_p)$ is the zero map. Up to powers of 2 which we can disregard since p is assumed to be odd, κ^* maps $\Sigma^{-3}u_{2r} \in H^{2r-3}(\Sigma^{-3}\mathbf{k})$ to

$$\pm\frac{B_r}{(2r)!}m(2r)(u_{-3}p_1^r).$$

(it is not hard to derive that this class is integral from Von Staudt's theorem and Lemma 2.1. of [37]). This reduces to a generator mod p if p does not divide $\frac{B_r}{(2r)!}m(2r)$ which is certainly the case if $p \geq 2r$ and p does not divide the numerator of B_r. $\square$

Proof of Theorem 6.0.5: Look at the diagram:

(6.0.6)
$$\begin{tikzcd}
H^1(\mathrm{MTSO}(3);\mathbb{F}_p) \arrow[d] & H^1(\mathrm{MTSO}(3);\mathbb{Z}) \arrow[l, two heads] \arrow[r] \arrow[d] & H^1(\mathrm{MTSO}(3);\mathbb{Q}) \arrow[d, "0"] \\
H^1(B\,\mathrm{Diff}^+(M);\mathbb{F}_p) & H^1(B\,\mathrm{Diff}^+(M);\mathbb{Z}) \arrow[l] \arrow[r, tail] & H^1(B\,\mathrm{Diff}^+(M);\mathbb{Q})
\end{tikzcd}$$

The right-hand vertical arrow is zero by Corollary 1.0.5 and the lower right horizontal arrow is injective ($H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Q})$ is injective for any space X). The upper left horizontal arrow is surjective since p is odd. An easy diagram chase shows that $H^1(\mathrm{MTSO}(3);\mathbb{F}_p) \to H^1(B\,\mathrm{Diff}^+(M);\mathbb{F}_p)$ is also trivial.

Let $\mathcal{A}_p$ denote the mod p-Steenrod algebra. Then the restriction of α to $\mathcal{A}_p \cdot H^1(\mathrm{MTSO}(3))$ is trivial because α is a spectrum map. Let $x := p_1 \in H^4(BSO(3);\mathbb{F}_p)$ be the first Pontrjagin class and let $\mathcal{P}$ be the total Steenrod power operation. We will compute $\mathcal{P}(\mathrm{th}_{-L_3}(x)) \in H^*(\mathrm{MTSO}(3);\mathbb{F}_p)$. This is done using a formula by Wu (compare [33], Thm 19.7). Let $r = \frac{1}{2}(p-1)$, $\mathcal{P}^i$ has degree $4ri$. Wu's formula states, in the present context, that

$$\mathcal{P}(\mathrm{th}_{-L_3}\,x) = \mathrm{th}_{-L_3}((x+x^p)(1+x^r)^{-1}).$$

Therefore $\mathcal{P}^i(\mathrm{th}_{-L_3}\,x) \in H^{1+4ri}(\mathrm{MTSO}(3);\mathbb{F}_p)$ agrees with $\mathrm{th}_{-L_3}(x^{ri+1})$, multiplied by the coefficient of z^{ri+1} in the power series

$$(z+z^p)(1+z^r)^{-1} = \sum_{l\geq 0}(-1)^l(z^{rl+1}+z^{rl+p}).$$

It is clear that this coefficient is a unit in $\mathbb{F}_p^\times$. Thus $\mathcal{P}^i(\mathrm{th}_{-L_3}\,x) \in H^{1+4ri}(\mathrm{MTSO}(3);\mathbb{F}_p)$ is a generator and we have argued above that $\alpha^*\mathcal{P}^i(\mathrm{th}_{-L_3}\,x) = 0$. This concludes the proof. $\square$

It appears to be quite difficult to find an example of a 3-manifold such that $\alpha : \Sigma^\infty B\,\mathrm{Diff}^+(M)_+ \to \mathrm{MTSO}(3)$ is nonzero in cohomology. Many computations which will not be reproduced here suggest that α could very well vanish in cohomology with arbitrary coefficients. However:

Proposition 6.0.7. *For $M = \mathbb{S}^3$, the MTW-map $\Sigma^\infty BSO(4)_+ \to \mathrm{MTSO}(3)$ of the universal $\mathbb{S}^3$-bundle is not nullhomotopic.*

Proof (The author owes this argument to O. Randal-Wiliams): Recall that α fits into the cofibre sequence 2.5.1:

$$\Sigma^\infty BSO(4)_+ \xrightarrow{\alpha} \mathrm{MTSO}(3) \xrightarrow{\eta} \Sigma\,\mathrm{MTSO}(4).$$

If α were nullhomotopic, then there exists a splitting $s : \Sigma\,\mathrm{MTSO}(4) \to \mathrm{MTSO}(3)$ of η (i.e. $s \circ \eta = \mathrm{id}$). In the sequel we assume that the map

$$\eta^* : H^*(\mathrm{MTSO}(4); \mathbb{F}_3) \to H^*(\Sigma^{-1}\,\mathrm{MTSO}(3); \mathbb{F}_3)$$

has a right inverse s^* as a map of $\mathcal{A}_3$-modules and show that this is impossible. Let u_{-3} be the Thom class of $-L_3$ and we write $u_{-3} \cdot x$ for $\mathrm{th}_{-L_3}(x)$ (recall that this is a module structure); similarly for MTSO(4). Since $\eta^* u_{-4} = \Sigma^{-1} u_3$, it follows that $s^* \Sigma^{-1} u_{-3} = u_{-4}$ and thus that

$$Qu_{-4} = \Sigma^{-1} s^* Q_{u_3} \tag{6.0.8}$$

for any $Q \in \mathcal{A}_3$. Put $Q = \mathcal{P}^3 - \mathcal{P}^2\mathcal{P}^1$. Recall the formulae

$$\mathcal{P}^1 p_1 = p_1^2 + p_2;\ \mathcal{P}^2 p_1 = p_1^3;\ \mathcal{P}^1 p_1^2 = -p_1(p_1^2 + p_2);\ \mathcal{P}^1 p_2 = p_1 p_2$$

for the Steenrod operations on $BSO(4)$ (and hence, by putting $p_2 = 0$, also on $BSO(3)$) and the formula

$$\mathcal{P}(u_4) = u_{-4}(K(p_1, p_2)),$$

where K is the multiplicative sequence associated with $(1+x)^{-1}$. Its lowest terms are $K(p_1, p_2) = 1 - p_1 + p_1^2 - p_1^3 - p_1 p_2 + \ldots$. From this, we get

$$(\mathcal{P}^3 - \mathcal{P}^2\mathcal{P}^1)(u_{-4}) = u_{-4} p_1 p_2$$

on MTSO(4). Therefore $(\mathcal{P}^3 - \mathcal{P}^2\mathcal{P}^1)(u_{-4}) = 0$ (just put $p_2 = 0$ and shift the degrees. This contradicts 6.0.8. □

We conclude this section by asking the question:

Question 6.0.9. Does there exist an oriented closed 3-manifold M and a prime p, such that $\alpha^*_{E_M} : \tilde{H}^*(\mathrm{MTSO}(3); \mathbb{F}_p) \to \tilde{H}^*(B\,\mathrm{Diff}^+(M); \mathbb{F}_p)$ is nontrivial?

7. What happens for manifold bundles with boundary?

In this section we study manifold bundles with boundary and ask whether the vanishing theorem 1.0.4 still holds for such bundles. We have to distinguish two cases. The first case is when we require the boundary bundle to be trivialized. In this case, Theorem 1.0.4, interpreted appropriately, is still true. The second case is when the boundary bundle is not required to be trivial. In this case the generalized MMM-classes are not defined in general and therefore the analogue of Theorem 1.0.4 does not make sense, as we will discuss briefly.

7.1. **Manifold bundles with boundary.** Let M be an n-dimensional (oriented, smooth, compact) manifold with boundary. There are two types of smooth M-bundles that come to mind.

We can study the structural group $\mathrm{Diff}^+(M)$ of all orientation-preserving diffeomorphisms, with no condition on the boundary. We say that a bundle with structural group $\mathrm{Diff}^+(M)$ and fibre M has *free boundary*. Or we can consider the group $\mathrm{Diff}^+(M;\partial)$ of diffeomorphisms of M that coincide with id on a small neighborhood of ∂M. Bundles with this structural groups are said to have *fixed boundary*.

7.2. **The Pontrjagin-Thom construction for bundles with boundary.** Let $f: E \to B$ be a manifold bundle with boundary $\partial f: \partial E \to B$ and of fibre dimension n. The isomorphism $T_v\partial E \oplus \mathbb{R} \cong T_vE|_{\partial E}$ defines a spectrum map $\eta_E: \mathbb{T}\mathbf{h}(-T_v\partial E) \to \Sigma\mathbb{T}\mathbf{h}(-T_vE)$ that fits into a commutative diagram (the rest of the diagram is explained below)

(7.2.1)
$$\begin{tikzcd}
\Sigma^\infty B_+ \arrow[d, "\mathrm{PT}_{\partial E}"] \arrow[dr, "\mathrm{PT}_E \simeq *"] & \\
\mathbb{T}\mathbf{h}(-T_v\partial E) \arrow[r, "\eta_E"] \arrow[d, "\kappa_{\partial E}"] & \Sigma\mathbb{T}\mathbf{h}(-T_vE) \arrow[d, "\kappa_E"] \\
\mathrm{MTSO}(n-1) \arrow[r, "\eta"] & \Sigma\,\mathrm{MTSO}(n) \arrow[d, "x"] \\
 & \Sigma A.
\end{tikzcd}$$

Choose an embedding $j: E \to B \times [0,\infty) \times \mathbb{R}^{\infty-1}$ such that $\partial E = j^{-1}(B \times \{0\} \times \mathbb{R}^{\infty-1})$ and $j(E) \subset B \times [0,1) \times \mathbb{R}^{k-1}$. The collapse construction defines a spectrum map

$$\mathrm{PT}_E : [0,\infty] \wedge \Sigma^{\infty-1} B_+ \to \mathbb{T}\mathbf{h}(-T_vE),$$

here $\infty \in [0,\infty]$ serves as a basepoint. If $t \gg 1$, then the composition

$$\Sigma^{\infty-1}B_+ \cong \{t\} \wedge \Sigma^{\infty-1}B_+ \to [0,\infty] \wedge \Sigma^{\infty-1}B_+ \stackrel{\mathrm{PT}_E}{\to} \mathbb{T}\mathbf{h}(-T_vE)$$

is the constant map. On the other hand, if $t = 0$, then

$$\Sigma^{\infty-1}B_+ \cong \{0\}_+ \wedge \Sigma^{\infty-1}B_+ \to [0,\infty] \wedge \Sigma^{\infty-1}B_+ \stackrel{\mathrm{PT}_E}{\to} \mathbb{T}\mathbf{h}(-T_vE)$$

is the composition $\eta_E \circ \mathrm{PT}_{\partial E} : \Sigma^{\infty-1}B_+ \to \Sigma^{-1}\mathbb{T}\mathbf{h}(-T_v\partial E) = \mathbb{T}\mathbf{h}(-T_vE|_{\partial E}) \to \mathbb{T}\mathbf{h}(-T_vE)$. In other words, the Pontrjagin-Thom map PT_E can be interpreted as a nullhomotopy of the composition $\eta_E \circ \mathrm{PT}_{\partial E}$, as displayed in diagram 7.2.1.

Let A be a spectrum, $x : \mathrm{MTSO}(n) \to A$ a map. By composing the nullhomotopy PT_E with $x \circ \kappa - E$, we get a nullhomotopy P of the composition $x \circ \kappa_E \circ \eta_E \circ \mathrm{PT}_{\partial E} : \Sigma^{\infty-1}B_+ \to \Sigma A$.

Suppose that there is a second nullhomotopy Q of the same map, but written as the composition

$$\Sigma^{\infty}B_+ \stackrel{\mathrm{PT}_\partial}{\to} \mathbb{T}\mathbf{h}(-T_v\partial E) \stackrel{\kappa_{\partial E}}{\to} \mathrm{MTSO}(n-1) \to \Sigma\,\mathrm{MTSO}(n-1) \to \Sigma A$$

(this is the same as $x \circ \eta \circ \alpha_{\partial E}$). Such a nullhomotopy typically arises from a vanishing theorem concerning the bundle ∂E and it only involves ∂E and some choices that do not depend on E. We can glue the two nullhomotopies Q and P and obtain a map $\Sigma^\infty B_+ \to A$. If the nullhomotopy Q is defined for the universal bundle $E_M \to B\,\mathrm{Diff}^+(M)$, we can use this procedure to define characteristic classes of smooth M-bundles. More precisely, even though the map $\alpha_{E_M} : \Sigma^\infty B\,\mathrm{Diff}^+(M)_+ \to \mathrm{MTSO}(n)$ does not exist, we can make sense out of the element $\alpha_{E_M}^*(x) \in A(B\,\mathrm{Diff}^+(M))$.

We list a few examples of situations to which the above philosophy can be applied. In each of these cases, we would need to make the map x as well as the nullhomotopy Q precise on the point-set level. We indicate how this works in the example that is of interest to us: the second example.

(1) If $\partial E = \emptyset$ and $x = \mathrm{id} : \mathrm{MTSO}(n) \to \mathrm{MTSO}(n)$ and Q is the constant nullhomotopy, then we get of course the MTW-map α_E back.
(2) (generalization of the first example) If $x = \mathrm{id}_{\mathrm{MTSO}(n)}$ and E has fixed boundary, then $\alpha_{\partial E} : \Sigma^\infty B_+ \to \mathrm{MTSO}(n-1)$ factors as $\Sigma^\infty B_+ \to \Sigma^\infty \mathbb{S}^0 \stackrel{\alpha_{\partial M}}{\to} \mathrm{MTSO}(n-1)$. But there is an oriented nullbordism W of ∂M (of course, $W = M$ is a possible choice, but there is no reason to prefer this choice). This nullbordism induces a nullhomotopy of the composition $\Sigma^\infty \mathbb{S}^0 \stackrel{\alpha_{\partial M}}{\to} \mathrm{MTSO}(n-1) \to \Sigma\,\mathrm{MTSO}(n)$. Thus we are in the above situation and hence we can define a map $\Sigma^\infty B_+ \to \mathrm{MTSO}(n)$, called α_E. Geometrically, this corresponds to gluing in the trivial bundle $B \times W$ into E along ∂E. If $\hat{E}$ denotes this new bundle, then $\alpha_E = \alpha_{\hat{E}}$. This geometric description, together with the homotopy equivalence $\alpha^{GMTW} : \Omega B\,\mathrm{Cob}_n \simeq \Omega^\infty\,\mathrm{MTSO}(n)$ from [20], explains how to the make the nullhomotopy precise. Note that this construction depends on the choice of W; thus α_E is well-defined only modulo maps of the form $\alpha_{B\times V}$ for constant bundles of closed manifolds. Therefore the map $\alpha_E : H^*(\mathrm{MTSO}(n)) \to H^*(B\,\mathrm{Diff}^*(M))$ does not depend on the choice of W as long as we consider positive degrees $* > 0$.
(3) If $A = \Sigma^k H\mathbb{Z}$, $y \in H^k(BSO(n))$ and χ is the Euler class, let $x = \mathrm{th}(y\chi) \in H^k(\mathrm{MTSO}(n))$. There is a canonical nullhomotopy of $x \circ \eta$, induced by a

nonzero cross-section of $L_n|_{BSO(n-1)}$. Thus we can apply the above construction. This shows that, although there is no map $\alpha_E : \Sigma^\infty B_+ \to \mathrm{MTSO}(n)$ for a bundle with boundary, we can still define what ought to be the pullback $\alpha_E^* \operatorname{th}(y\chi)$, which should be the same as $f_!(y(T_vE)\chi(T_vE))$. Recall that the Becker-Gottlieb transfer $\operatorname{trf}_f : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ also exists when $f : E \to B$ is a manifold bundle with boundary. One can show that $\operatorname{trf}_f^*(y) = \alpha_E^* \operatorname{th}(y\chi)$.

(4) If n is even and $x : \mathrm{MTSO}(n) \to K$ is the universal symbol class of the signature operator, the vanishing theorem implies that we can define the index of the signature on an arbitrary bundle of even dimension. On the other hand, for odd n, it turns out that the index of the signature of the boundary is an obstruction to define the index of the odd signature operator.

We can use the second example from above to define the MTW-map of any bundle with fixed boundary. Since, in the above notation, α_E is defined to be the MTW-map of the closed bundle $\hat{E}$, Theorem 4.3.1 is still true for the new MTW-map of a bundle with boundary. All consequences that were derived from 4.3.1 by formal computations are still true, namely Theorems 1.0.4, 6.0.4, 6.0.5.

References

[1] J. F. Adams: *Chern characters revisited*. Illinois Journal of Mathematics 17 (1973), 333-336.

[2] J. F. Adams: *On Chern characters and the structure of the unitary groups*. Proceed. of the Cambridge Philosophical Society 57 (1961), 189-199.

[3] J. F. Adams: *Stable homotopy and generalised homology*. Reprint of the 1974 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, (1995).

[4] D. W. Anderson: *The real K-Theory of classifying spaces*. Proc. National Acad. Sci. U.S.A. 51 (1964), 634-636.

[5] M. F. Atiyah: *The signature of fibre bundles*. Global Analysis (Papers in Honor of K. Kodaira), pp. 73-84. Univ. Tokyo Press, Tokyo (1969).

[6] M. F. Atiyah; V. K. Patodi; I. M. Singer: *Spectral asymmetry and Riemannian geometry III*. Math. Proc. Cambridge Philos. Soc 79 (1976), 71-99.

[7] M. F. Atiyah; G. Segal: *Twisted K-theory*. Ukr. Mat. Visn. 1 (2004), 287-330.

[8] M. F. Atiyah; G. Segal: *Equivariant K-Theory and completion*. J. Differential Geometry 3 (1969).

[9] M. F. Atiyah; I. M. Singer: *The index of elliptic operators III*. Ann. of Math. 87 (1968), 546-604.

[10] M. F. Atiyah; I. M. Singer: *The index of elliptic operators IV*. Ann. of Math. 93 (1971), 119-138.

[11] M. F. Atiyah; I. M. Singer: *The index of elliptic operators V*. Ann. of Math. 93 (1971), 139-149.

[12] M. F. Atiyah; I. M. Singer: *Index theory for skew-adjoint Fredholm operators*. Publ. Math. IHES 37 (1969), 5-26.

[13] J. C. Becker; D. H. Gottlieb: *The transfer map and fiber bundles*. Topology 14 (1975), 1-12.

[14] U. Bunke: *Index Theory, Eta Forms, and Deligne Cohomology*. Memoirs of the American Mathematical Society 198 (2009).

[15] C. Douglas: *On the fibrewise Poincaré-Hopf Theorem*. Recent developments in algebraic topology, 101-111, Contemp. Math. 407, A.M.S., Providence, RI, 2006.

[16] J. Ebert: *Algebraic independence of generalized Morita-Miller-Mumford classes*. Preprint, arXiv:0910.1030 (2009).

[17] D. Freed; M. Hopkins; C. Teleman: *Loop groups and twisted K-theory I*. Preprint, arXiv:0711.1906, 2007.

[18] S. Galatius: *Mod 2 homology of the stable spin mapping class group.* Math. Ann. 334 (2006), 439-455.

[19] S. Galatius; I. Madsen, U. Tillmann: *Divisibility of the stable Miller-Morita-Mumford classes*, J. of the A.M.S., 19 (2006), 759-779.

[20] S. Galatius; I. Madsen; U. Tillmann; M. Weiss: *The homotopy type of the cobordism category.* Acta Math. 202, (2009), 195-239.

[21] J. Giansiracusa, U. Tillmann: *Vanishing of universal characteristic classes for handlebody groups and boundary bundles.* Preprint, arXiv:0910.5367 (2009).

[22] A. Hatcher; N. Wahl: *Stabilization for mapping class groups of 3-manifolds.* Preprint, ArXiv:0709.2173 (2007).

[23] F. Hirzebruch: *Neue topologische Methoden in der algebraischen Geometrie.* Springer-Verlag, (1962).

[24] M. Karoubi: *K-Theory. An Introduction.* Reprint of the 1978 edition, Springer-Verlag 2008.

[25] M. A. Kervaire: *Relative characteristic classes.* Amer. J. Math. 79 (1957), 517-558.

[26] N. H. Kuiper: *The homotopy type of the unitary group of Hilbert space.* Topology 3 (1965), 19-30.

[27] W. Lück; A. Ranicki: *Surgery obstructions in fiber bundles.* J. Pure Appl. Algebra 81 (1992), 139-189.

[28] H. B. Lawson; M. Michelsohn: *Spin geometry.* Princeton University Press, (1989).

[29] I. Madsen; U. Tillmann: *The stable mapping class group and* $\mathbb{CP}^{\infty}_{-1}$. Invent. Math. 145 (2001), 509-544.

[30] I. Madsen; M. Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture.* Ann. of Math. 165 (2007), 843-941.

[31] W. Meyer: *Die Signatur von Faserbündeln und lokalen Koeffizientensystemen.* Bonner Mathematische Schriften 53 (1972).

[32] J. W. Milnor; J. C. Moore: *On the structure of Hopf algebras.* Ann. of Math. 81 (1965), 211-264.

[33] J. W. Milnor; J. Stasheff: *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J. (1974).

[34] E. Y. Miller: *The homology of the mapping class group.* J. Differential Geom. 24 (1986), 1-14.

[35] S. Morita: *Characteristic classes of surface bundles.* Invent. Math. 90 (1987), 551-577.

[36] D. Mumford: *Towards an enumerative geometry of the moduli space of curves.* Arithmetic and geometry, Vol. II, 271-328, Progr. Math., 36, Birkhüser Boston, Boston, MA, 1983.

[37] G. Pappas: *Integral Riemann-Roch theorem.* Invent. Math. 170 (2007), 455-481.

[38] B. L. Reinhart: *Cobordism and the Euler number.* Topology 2 (1963) 173-177.

[39] Y. B. Rudyak: *On Thom spectra, orientability and cobordism.* Springer-Verlag, 1998.

[40] U. Tillmann: *On the homotopy of the stable mapping class group.* Invent. Math. 130 (1997), 257-275.

Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Bundesrepublik Deutschland

E-mail address: ebert@math.uni-bonn.de